THE LIFTING FORCE ACTION ON A CONTOUR IN A PLANE HOMOGENEOUS VORTEX FLOW OF AN INCOMPRESSIBLE IDEAL FLUID

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ABSTRACT: The plane problem of homogeneous vortex flows of an incompressible inviscid fluid around a contour is considered. A method is developed for contours having a center or an axis of symmetry for calculating the lifting force acting on the contour which reduces the problem of determining the lifting force to an ordinary problem of a potential flow of a fluid around the given contour.

A fixed contour is placed in a plane, infinite, homogeneous, vortex flow of an incompressible, inviscid fluid (Fig. 1). As shown by the solution for the problem of a circle [1], in this case, unlike potential flows around a contour, a lifting force proportional to the vortex is set up.

The problem of determining lifting forces is solved below for a certain class of contours.

In a Cartesian coordinate system, the velocity of a fluid at infinity is

$$u = u_\infty + \omega y, \quad v = 0 \quad \text{when } r \to \infty \;\; (r = \sqrt{x^2 + y^2}) \,. \tag{1}$$

Here u, v are projections of the velocity vector on the x, y axes, respectively; u_∞ is the velocity at infinity as $x \to \infty$, $y = 0$. According to the Helmholtz theorem [2], in this case, the vorticity will be constant over the entire flow region and equal to $-\omega$.

Thus, flow around a contour is described by the equations with boundary conditions (1) at infinity and the condition on the contour

$$\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} = -\omega, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

with boundary conditions (1) at infinity and the condition on the contour

$$w_n = u\cos(n, x) + v\sin(n, x) = 0 \,. \tag{3}$$

Here n is the direction of the outward normal from the contour and w_n is the normal component of the velocity along the contour.

In order to calculate the forces acting on the contour, it is sufficient to know the asymptotic behavior of the velocity and the pressure in the vicinity of an infinitely remote point.

We shall seek the solution of system (2) in the form

$$u = U + u_1 + \omega y, \qquad v = V + v_1 \,. \tag{4}$$

In this case, U and V satisfy the equations

$$\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0, \qquad \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} = 0, \tag{5}$$

and the boundary conditions

$$U\cos(n, x) + V\sin(n, x) = 0 \text{ on } L;$$
$$U = u_\infty, \quad V = 0 \text{ at } \infty \,. \tag{6}$$

The functions u_1, v_1 satisfy the same equations (5) and the boundary conditions

$$u_1\cos(n, x) + v_1\sin(n, x) = -y\omega\cos(n, x) \text{ on } L;$$
$$u_1 = 0, \quad v_1 = 0 \text{ at } \infty \,. \tag{7}$$

After solving (5) with the corresponding boundary conditions (6) and (7) and substituting the solutions in (4), we obtain the solution of system (2) with conditions (1) and (3).

System (5) with conditions (6) describes the potential of a uniform flow around the given contour with velocity u_∞ at infinity. In complex variables the total conjugate velocity of such a flow in the vicinity of an infinitely remote point without circulation is represented by the expansion

$$\overline{w} = \overline{u}_\infty + A_2 / z^2 + A_3 / z^3 + \ldots,$$
$$(A_k = A_k' + iA_k'') \quad (k = 2, 3, 4, \ldots). \tag{8}$$

System (5) with conditions (7) also describes a certain potential flow with zero velocity at infinity. The solution will be of the form

$$\overline{w}_1 = B_2 / z^2 + B_3 / z^3 + \ldots \qquad (B_k = B_k' + iB_k'') \,. \tag{9}$$

The complex-conjugate velocity of the total flow will be

$$\overline{w}_0 = \overline{w} + \overline{w}_1 = \overline{u}_\infty + C_2 / z^2 + C_3 / z^3 + \ldots,$$
$$(C_k = A_k + B_k). \tag{10}$$

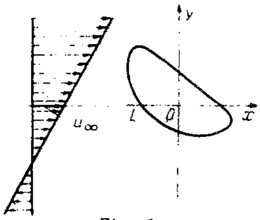

Fig. 1

Since $w_0 = u_0 - iv_0$ and $z = re^{i\varphi}$, we obtain the following expansion in the vicinity of an infinitely remote point:

$$u = u_\infty + \omega r\sin\varphi + \frac{C_2'\cos 2\varphi + C_2''\sin 2\varphi}{r^2} + O\left(\frac{1}{r^3}\right),$$
$$v = \frac{C_2'\sin 2\varphi - C_2''\cos 2\varphi}{r^2} + O\left(\frac{1}{r^3}\right). \tag{11}$$

The pressure is determined from the Euler equations. In polar coordinates

$$\frac{\partial p}{\partial r} = -\frac{2C_2'\rho\omega}{r^2}\sin^3\varphi - \frac{C_2 2\rho\omega}{r^2}[2\cos^2\varphi - 3\cos\varphi] + O\left(\frac{1}{r^3}\right),$$
$$\frac{\partial p}{\partial\varphi} = \frac{6C_2'\rho\omega}{r}[\cos\varphi - \cos^3\varphi] + \frac{3C_2''\rho\omega}{r}[2\sin^2\varphi - 1] + O\left(\frac{1}{r^2}\right), \tag{12}$$

Integrating these equations with $p = p_0$ at infinity, we obtain

$$p = p_0 + \frac{C_2' 2\rho\omega}{r}\sin^3\varphi + \frac{C_2''\rho\omega}{r}[2\cos^3\varphi - 3\cos\varphi] + O\left(\frac{1}{r^2}\right) \tag{13}$$

We also compute vv_r, where $v_r = u\cos\varphi + v\sin\varphi$ is the radial velocity,

$$vv_r = \frac{\omega}{r}\left[\frac{C_2'\sin^2 2\varphi}{2} - \frac{C_2''\cos 4\varphi}{4}\right] + O\left(\frac{1}{r^2}\right). \tag{14}$$

The momentum equation for the region bounded by the given contour and a circle of arbitrary radius R is of the form

$$\int_0^{2\pi} \rho v_r v R\, d\varphi + \int_0^{2\pi} PR\sin\varphi\, d\varphi + F = 0 \tag{15}$$

where F is the force acting on the contour in the direction of the y axis, that is, the lifting force. Substituting (13) and (14) here, then integrating, we obtain

$$F = 2\pi C_2'\rho\omega \,. \tag{16}$$

It follows from the last relationship that the lifting force does not depend on C_2'', or, consequently, on A_2'' or B_2'', that is, on the imaginary parts of the coefficients A_2 and B_2 in expansions (8) and (9).

Thus, calculating the lifting force is reduced to determining the real parts of coefficients A_2 and B_2.

The quantity A_2 can be found if we know the ordinary flow around a contour which is uniform at infinity. In particular, if we know the conformal mapping of the exterior of a circle of radius R onto the exterior of the given contour $z = f(\xi)$, then

$$A_2 = kk_1u_\infty - k^2R^2u_\infty. \tag{17}$$

Here k, k_1 are the coefficients of the Laurent expansion of the function

$$z = f(\xi) = k\xi + k_0 + \frac{k_1}{\xi} + \frac{k_2}{\xi^2} + \ldots. \tag{18}$$

As was pointed out previously, the value of B_2' can also be determined if we know the solution of system (5) with conditions (7).

We shall show that with proper choice of the position of the coordinate system relative to the contour, or more precisely, the position of the x axis which corresponds to choice of a certain value of the velocity at infinity, we can ensure that the magnitude of the coefficient B_2' vanish in expansion (9). Then the problem of determining the lifting force is reduced to the thoroughly studied problem of a potential flow around a contour.

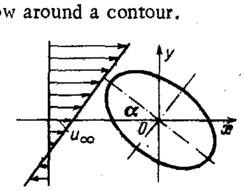

Fig. 2

Indeed, translating the x axis the distance a is equivalent to changing u_∞ by the amount $a\omega$ and replacing the boundary conditions (7) by

$$u_1 \cos(n, x) + v_1 \sin(n, x) =$$
$$= -y\omega\cos(n, x) - a\omega\cos(n, x). \tag{19}$$

Then, u_1 and v_1 of (7) can be represented in the form of a sum

$$u_1 = u_1^* + u_1^{**}, \quad v_1 = v_1^* + v_1^{**},$$

where u_1^* and v_1^* satisfy equations (5) and conditions (7), while u_1^{**} and v_1^{**} satisfy the same equations and the following boundary conditions:

$$u_1^{**}\cos(n, x) + v_1^{**}\sin(n, x) =$$
$$= -a\omega\cos(n, x) \text{ on } L u_1^{**} = 0, v_1^{**} = 0 \text{ at } \infty. \tag{20}$$

Consequently, u_1^* and v_1^* constitute a solution of (5) and (7) in the "old coordinates," while u_1^{**} and v_1^{**}, as can be readily seen, describe a flow which is a superposition of a homogeneous flow of a fluid with velocities $u = a\omega$, $v = 0$ and potential flow around the given contour of a fluid with the velocity $u = -a\omega$ at infinity. In other words, for the velocity complex-conjugate to $w_1^{**} = u_1^{**} + iv_1^{**}$, we can write

$$\bar{w}_1^{**} = u_1^{**} - iv_1^{**} = D_2/z^2 + D_3/z^3 + \ldots,$$
$$(D_k = D_k' + iD_k''),$$

where, according to (17),

$$D_2 = kk_1a\omega - k^2R^2a\omega. \tag{21}$$

Thus, D_2, D_2', D_2'' are proportional to the quantity a. By proper choice of a, we can achieve the equality $B_2' = D_2'$. Then, in the new coordinate system, the real part of the coefficient B_2 will be equal to zero and, consequently, $C_2' = A_2'$, that is, in order to calculate the lifting force, it will be sufficient to know the ordinary flow around the contour and substitute the value of the velocity corresponding to the chosen axis into (16) instead of u_∞.

In this case, it is easy to obtain a second expression for the lifting force. The quantity A_2' [3] is expressed by the area under the contour and the apparent additional mass corresponding to the direction of the velocity at infinity

$$A_2' = \frac{1}{2\pi}\left(\frac{m}{\rho} + S_0\right)u_\infty. \tag{22}$$

Here m is the apparent additional mass, S_0 is the area under the contour. Then, we shall have for the lifting force

$$F = (m + \rho S_0)\omega u_\infty. \tag{23}$$

It is necessary to emphasize again that the unknown u_∞ in the last relationship depends on a. In the general case, a is determined from the solution of problem (5) and (7).

However, we can point out two classes of contours for which it is not difficult to select the position of the x axis.

In the first place, these are contours which do not possess axial symmetry, being arranged so that the velocity of whirling flow at infinity is parallel to the axis of symmetry. In this case, the x axis should coincide with the axis of symmetry. We shall show this. It follows from the boundary conditions (7) in this case that the following should be true at the conjugate points of the region, that is, when $z = r\exp(\pm i\varphi)$:

$$u_1' = -u_1'', \quad v_1' = v_1''. \tag{24}$$

Since
$$u_1 = \frac{B_2'\cos 2\varphi + B_2''\sin 2\varphi}{r^2} + O\left(\frac{1}{r^3}\right),$$
$$v_1 = \frac{B_2'\sin 2\varphi - B_2''\cos 2\varphi}{r^2} + O\left(\frac{1}{r^3}\right), \tag{25}$$

B_2' should be equal to zero in order to satisfy (24).

In the second place, these are contours which possess symmetry relative to a point (central symmetry). In the case of such contours, we must have the axis pass through the center of symmetry. In this case, the boundary conditions (7) are such that we should have

$$\bar{w}_1(z) = -\bar{w}_1(-z). \tag{26}$$

However, on the other hand,

$$\bar{w}_1(z) = \frac{B_2}{z^2} + \frac{B_3}{z^3} + \frac{B_4}{z^4} + \ldots,$$
$$\bar{w}_1(-z) = \frac{B_2}{z^2} - \frac{B_3}{z^3} + \frac{B_4}{z^4} - \ldots. \tag{27}$$

It is clear from comparing (27) and (26) that all B_k with even subscripts should be zero. Thus, in the given case, not only B_2', but B_2'', is equal to zero.

Now, we shall present an example of calculation of the lifting force. An ellipse with semiaxes a, b (a > b) is placed in a uniformly whirling flow at an angle of attack α (Fig. 2). Since the ellipse possesses central symmetry, the x-axis passes through its center.

The complex potential Φ of the potential flow around the ellipse with velocity u_∞ is of the form [2]

$$\Phi = \frac{1}{a-b}\left[(az - b\sqrt{z^2 - c^2})\,u + i\,(bz - a\sqrt{z^2 - c^2})\,v\right]e^{-\alpha i}.$$

Here

$$(u = u_\infty\cos\alpha,\ v = u_\infty\sin\alpha;\ c^2 = a^2 - b^2).$$

Since in the vicinity of an infinitely distant point

$$\sqrt{z^2 - c^2} = z - \frac{c^2}{2z} + \ldots,$$

in this same neighborhood

$$\Phi = e^{-i\alpha}\left[(u - iv)\,z + \frac{a+b}{2}(bu + iav)\frac{1}{z} + \ldots\right].$$

Hence

$$A_2 = -\tfrac{1}{2}(a-b)(bu + iav)e^{-\alpha i}.$$

On separating the real part, we obtain

$$A_2' = u_\infty(a+b)\left[b\cos^2\alpha + a\sin^2\alpha\right].$$

The lifting force is equal to

$$F = \pi (a + b) [b \cos^2 \alpha + a \sin^2 \alpha].$$

Setting a = b = R in the last expression, we have

$$F = 2\pi R^2 \rho \omega u_\infty,$$

This coincides [1] with the known expression for the lifting force for a circle.

If we set b = 0, $0 \le \alpha \le \pi/2$, we have

$$F = \pi a^2 \sin^2 \alpha \rho \omega u_\infty,$$

that is, the plate at an angle of attack has a lifting force; this force reaches a maximum at $\alpha = \pi/2$ when the plate is perpendicular to the flow. This force is of the same origin as the suction force in the ordinary case of a profile with sharp leading and trailing edges.

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